

WELL-POSEDNESS OF EINSTEIN'S EQUATION WITH REDSHIFT DATA

CHRISTOPHER J. WINFIELD

ABSTRACT. We study the solvability of a system of ordinary differential equations derived from null geodesics of the LTB metric with data given in terms of a so-called redshift parameter. Data is introduced along these geodesics by the luminosity distance function. We check our results with luminosity distance depending on the cosmological constant and with the well-known FRW model.

INTRODUCTION

Resulting from the Lemaître-Tolman-Bondi metric

$$(0.1) \quad ds^2 = -dt^2 + \frac{R'(t,r)^2 dr^2}{1 + 2E(r)} + R(t,r)^2 d\Omega^2$$

1991 *Mathematics Subject Classification.* 83F05, 34A34.

Key words and phrases. redshift parameter, ordinary differential equations, LTB cosmology, luminosity distance.

The author thanks the Mathematics Departments of UW-Madison and UW-Oshkosh and the UW-Madison Physics Department for the use of their resources during the course of the present research. The author would particularly like to thank Prof. Daniel Chung of the UW-Madison Physics Department for his conversations and suggestions leading to the present article.

are so-called symmetric dust solutions to the Einstein equation given by

$$(0.2) \quad \left(\frac{\dot{R}}{R} \right)^2 = \frac{2E}{R^2} + \frac{2M}{R^3}$$

$$(0.3) \quad \rho(t, r) = \frac{M'(r)}{R(t, r)^2 R'(t, r)}$$

(c.f. [17]) for some suitable ρ (energy density) where superscript \cdot and \cdot denote partial derivatives with respect to r and t , respectively. Setting $\sigma \stackrel{\text{def}}{=} \text{sgn} \dot{R}$, $\delta \stackrel{\text{def}}{=} \text{sgn} R'$, $A \stackrel{\text{def}}{=} B\sqrt{1+2E}$ and $B \stackrel{\text{def}}{=} \sigma\sqrt{2E+2M/R}$, we study resulting system [6, 3]

$$(0.4) \quad \frac{dr}{dz} = \frac{\sqrt{1+2E}}{(1+z)\partial_{r,t}^2 R(t, r)} = \frac{A}{E' + M'/R - MR'/R^2}$$

$$(0.5) \quad \frac{dt}{dz} = \frac{-|R'|}{(1+z)\partial_{r,t}^2 R(t, r)} = \frac{-BR'\delta}{E' + M'/R - MR'/R^2},$$

taken along null geodesics of (0.1). Here data is given for the function R , prescribing values $R(t(z), r(z))$ along curves given by (0.4) and (0.5). As a result, corresponding solutions of this system provide maps

$$(0.6) \quad (E(r), D_L(z), R_0(r)) \rightarrow (r(z), t(z), M(r(z)))$$

as introduced in [6] which we study in some detail in this article.

As an application of our analysis, we will consider data given in the form

$$(0.7) \quad R(t(z), r(z)) = \frac{D_L(z)}{(1+z)^2} = \frac{\int_1^{1+z} \mathcal{I}(y) dy}{1+z}$$

for $D_L(z) = (1+z) \int_1^{1+z} \mathcal{I}(y) dy$ with $\mathcal{I}(y) = 1/\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)y^3}$ for a real parameter $0 \leq \Omega_\Lambda \leq 1$. Here, D_L is generally referred to as "luminosity distance"

and, in particular models, $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ is directly proportional to the so-called "cosmological constant" Λ [3, 5]. For further details of the physical and mathematical derivation of the present problem, the author recommends the aforementioned articles along with [7, 9, 10, 15, 19, 14] - to name but a few.

This work is physically motivated by competing cosmological theories in explaining certain observations of matter distribution and cosmic inflation. Such theories include those of "dark energy" [8], certain metric perturbations from the FRW model [11, 16, 13], radial inhomogeneities of the unperturbed LTB model (via $E(r)$, $R_0(r)$ and $M(r)$), and the cosmological constant (here via D_L) - with our work involving the later two. Here, we study the map (0.6) mostly on purely mathematical grounds, presenting a framework of analysis and, in a special case, estimates on resulting functions M in terms of z . Furthermore, we test our results for certain functions D_L , E , and R_0 , arising from various FRW-type models, and study singularities of M as indications of (in-) compatibility of these models.

1. SINGULARITIES

From the Chain Rule (c.f. equation (14) [6]) we observe that R' takes the form

$$R' = \mathcal{F}(R, R_0, R'_0, E, E', M, t) + M' \mathcal{G}(R, R_0, E, E', M).$$

With $\dot{R} = \sigma\sqrt{2E + 2M/R}$ and $R_0(r) \stackrel{\text{def}}{=} R(r, t_0)$ for a fixed $t_0 > 0$, we restrict $R, R_0, t > 0$, $M \geq -ER$, $E > 0$ and set

$$(1.1) \quad J(R, R_0, M, E, t) \stackrel{\text{def}}{=} \sqrt{2}(t - t_0) - \sigma \int_{R_0}^R \sqrt{\frac{\tau}{\tau E + M}} d\tau = 0$$

solutions of which define smooth manifolds \mathfrak{D}^\pm depending on constant $\sigma = \pm 1$, respectively.

We introduce notation: For a given function $f = f(t, r)$, depending implicitly or explicitly on (t, r) , we will denote $f[z] \stackrel{\text{def}}{=} f(t(z), r(z))$ and, with slight abuse of notation, set $\frac{df}{dz} \stackrel{\text{def}}{=} \frac{df[z]}{dz}$. We now set

$$(1.2) \quad R' = \mathcal{F} + \mathcal{G} \frac{dM/dz}{dr/dz}$$

where, from the chain rule, with subscript denoting the associated partial derivative,

$$(1.3) \quad -(\partial_R J)\mathcal{F} = E' \partial_E J + R'_0 \partial_{R_0} J$$

$$(1.4) \quad -(\partial_R J)\mathcal{G} = \partial_M J.$$

with $\xi \stackrel{\text{def}}{=} M/R$, $\xi^\sharp \stackrel{\text{def}}{=} M/R_0$, $\mathfrak{h} \stackrel{\text{def}}{=} R_0/R$, and

$$(1.5) \quad \begin{aligned} J_R &= -\sigma/\sqrt{E + \xi} & J_{R_0} &= \sigma/\sqrt{E + \xi^\sharp} \\ J_M &= -\frac{\sigma}{2} \int_1^{\mathfrak{h}} \frac{\nu^{1/2} d\nu}{(E\nu + \xi)^{3/2}} & J_E &= -\frac{\sigma R}{2} \int_1^{\mathfrak{h}} \frac{\nu^{3/2} d\nu}{(E\nu + \xi)^{3/2}} \end{aligned}$$

Substituting (1.2) into equations (0.4) and (0.5), we obtain

$$(1.6) \quad \left(E' - \frac{M}{R^2} \mathcal{F} \right) \cdot \frac{dr}{dz} + \frac{\frac{dM}{dz}}{R} - \frac{M \frac{dM}{dz}}{R^2} \mathcal{G} = A$$

$$(1.7) \quad \begin{aligned} & \left(E' - \frac{M}{R^2} \mathcal{F} \right) \cdot \frac{dr}{dz} \frac{dt}{dz} + \left(\frac{\frac{dM}{dz}}{R} - \frac{M \frac{dM}{dz}}{R^2} \mathcal{G} \right) \frac{dt}{dz} \\ &= -\delta \cdot B \cdot \left(\mathcal{F} + \frac{dM/dz}{dr/dz} \mathcal{G} \right) \frac{dr}{dz} \end{aligned}$$

We then substitute

$$\left(E' - \frac{M}{R^2} \mathcal{F} \right) \frac{dr}{dz} = A - \left(\frac{\frac{dM}{dz}}{R} - \frac{M \frac{dM}{dz}}{R^2} \mathcal{G} \right)$$

so that equation (1.7) becomes

$$(1.8) \quad A \frac{dt}{dz} = -\delta \cdot B \cdot \left(\mathcal{F} \frac{dr}{dz} + \frac{dM}{dz} \mathcal{G} \right).$$

Equation (1.8) can be verified by equations (0.2) and (0.3).

Now, equations (1.6), (1.8), and (1.2) along with the Chain Rule result in the following system:

$$\begin{aligned} \left(E' - \frac{M}{R^2} \mathcal{F} \right) \frac{dr}{dz} + \left(\frac{1}{R} - \frac{M \mathcal{G}}{R^2} \right) \frac{dM}{dz} &= A \\ \delta B \mathcal{F} \frac{dr}{dz} + A \frac{dt}{dz} + \delta B \mathcal{G} \frac{dM}{dz} &= 0 \\ \mathcal{F} \frac{dr}{dz} + \sigma \sqrt{(2E + 2M/R)} \frac{dt}{dz} + \mathcal{G} \frac{dM}{dz} &= \frac{dR}{dz} \end{aligned}$$

which we may write in matrix form as

$$(1.9) \quad \mathcal{U} \frac{d\vec{X}}{dz} = \vec{Y}$$

for

$$\mathcal{U} \stackrel{\text{def}}{=} \begin{pmatrix} E' - \frac{M}{R^2}\mathcal{F} & 0 & \frac{1}{R} - \frac{M\mathcal{G}}{R^2} \\ \delta B\mathcal{F} & A & \delta\mathcal{G}B \\ \mathcal{F} & \sigma\sqrt{2E + 2M/R} & \mathcal{G} \end{pmatrix}$$

$$\vec{X} = \begin{pmatrix} r \\ t \\ M \end{pmatrix}, \vec{Y} = \begin{pmatrix} A \\ 0 \\ \frac{dR}{dz} \end{pmatrix}.$$

We check the invertibility of \mathcal{U} as we compute

$$\begin{aligned} \det \mathcal{U} &= (E' - \frac{M}{R^2}\mathcal{F})(A\mathcal{G} - \sigma\delta\mathcal{G}B\sqrt{2E + 2M/R}) \\ &\quad + (\frac{1}{R} - \frac{M\mathcal{G}}{R^2})(\delta\sigma B\mathcal{F}\sqrt{2E + 2M/R} - \mathcal{F}A) \\ &= B(E'\mathcal{G} - \frac{\mathcal{F}}{R})(\sqrt{1 + 2E} - \sigma\delta\sqrt{2E + 2M/R}) \end{aligned}$$

From these computations we conclude

Proposition 1.10. *Suppose that $E, R > 0$ with $\partial_t R, \partial_r R, \partial_{t,r}^2 R \neq 0$. Then, \mathcal{U}^{-1} is a smooth function of R, R_0, R'_0, E, E' , and M except for the following cases:*

Either

- 1.) *both $\delta = \sigma$ and $R = 2M$; or,*
- 2.) *$E' R \mathcal{G} = \mathcal{F}$.*

We may extend the domain of \mathcal{U} to include $-1/2 < E < 0$, say, but for simplicity we impose the above hypothesis throughout the rest of this section.

We continue with

Proposition 1.11. *Suppose that for some $z^* > 0$, $\delta[z^*] = \sigma[z^*]$ and that*

$\frac{dR[z]}{dz}|_{z=z^} = 0$. Then, the matrix $\mathcal{U}[z]$ is singular at $z = z^*$.*

Proof. We have from the Chain Rule and equations (0.4) and (0.5) that

$$\begin{aligned}
 (1.12) \quad \frac{dR}{dz} &= R' \frac{dr}{dz} + \dot{R} \frac{dt}{dz} \\
 &= R' \frac{\sqrt{1+2E}}{(1+z)\partial_{r,t}^2 R} - \dot{R} \frac{|R'|}{(1+z)\partial_{r,t}^2 R} \\
 &= R' \frac{\sqrt{1+2E} - \delta\sigma\sqrt{2E+2M/R}}{(1+z)\partial_{r,t}^2 R}
 \end{aligned}$$

By our hypotheses on the partial derivatives of R we may conclude

$$(\sigma\delta\sqrt{2E+2M/R})[z^*] = (\sqrt{1+2E})[z^*]$$

With $\sigma = \delta$ at $z = z^*$, we have that $\sigma\delta = \delta^2 = 1$ and that $\sqrt{2E+2M/R} = \sqrt{1+2E}$ so that $2M[z^*] = R[z^*]$. Then from Proposition 1.10 we see that $\det U[z^*] = 0$. \square

We note that the type of singularity of item 1) of Proposition 1.10 appears analogous to that of the well-known "Schwarzschild" singularity: It is not yet clear here if this is merely an artifact of the specific model or if such singularities are removable by passing to alternate coordinate systems or metrics (c.f. §31 [14], §6.4 [18]), taking us beyond the scope of the present article.

We may interpret item 2) of Proposition 1.10 in terms of the tangent bundles $T\mathfrak{D}^\pm$ (resp.) of manifolds obtained from (1.1). We may consider the transformation $\phi^\pm : \mathbb{R} \times (0, +\infty) \rightarrow \mathfrak{D}^\pm$ given by $\phi^\pm(t, r) \stackrel{\text{def}}{=} (R(t, r), R_0(r), E(r), M(r), t)$

and $d\phi$ as a push forward, to interpret corresponding solutions to

$$E'R\partial_M J = E'\partial_E J + R'_0\partial_{R_0} J$$

as subsets \mathcal{M}^\pm of $T\mathfrak{D}^\pm$ (resp.) in coordinate form. Let \mathcal{T} denote the set $(\Omega^+ \setminus \pi\mathcal{M}^+) \cup (\Omega^- \setminus \pi\mathcal{M}^-)$ where π denotes the natural projection $\pi : T\mathcal{M} \rightarrow \mathcal{M}$ of a manifold \mathcal{M} .

By calculating $\mathcal{U}^{-1}\vec{Y}$ from (1.9) with $R_z \stackrel{\text{def}}{=} \frac{dR[z]}{dz}$, we arrive at the following system of ordinary differential equations:

$$(1.13) \quad \begin{aligned} \frac{dr}{dz} &= \frac{\mathcal{G}AR}{\mathcal{G}E'R - \mathcal{F}} - \frac{R_z \cdot (M\mathcal{G} - R)\sqrt{1+2E}}{R \cdot (\delta\sigma\sqrt{2E+2M/R} - \sqrt{1+2E})(\mathcal{G}E'R - \mathcal{F})} \\ \frac{dt}{dz} &= \frac{\delta \cdot R_z}{\delta\sigma\sqrt{2E+2M/R} - \sqrt{1+2E}} \\ \frac{dM}{dz} &= \frac{-\mathcal{F}RA}{\mathcal{G}E'R - \mathcal{F}} - \frac{R_z \cdot (R^2E' - \mathcal{F}M)\sqrt{1+2E}}{R \cdot (\delta\sigma\sqrt{2E+2M/R} - \sqrt{1+2E})(\mathcal{G}E'R - \mathcal{F})} \end{aligned}$$

We are ready to state

Proposition 1.14. *The matrix \mathcal{U} is non-singular for $R \neq 2M$ provided $(R, R_0, E, M, t) \blacksquare$*

$\in \mathcal{T}$. Indeed, if for some $z_0 > 0$, these conditions hold for $(R, R_0, E, M, t)[z]|_{z=z_0}$, then the system of equations (1.13) has a unique C^∞ solution $\vec{X}[z]$ in some open interval containing z_0 .

Proof. It is clear that the elements of U are continuously differentiable where $\det U$ is non-zero. The result follows by applying standard theory of ordinary differential equations [4]. \square

To further investigate the solvability of the system (1.13), we compute

$$\begin{aligned}
 (1.15) \quad \mathcal{G}E'R - \mathcal{F} &= \frac{E'R \cdot (J_E/R - J_M) + R'_0 J_{R_0}}{J_R} \\
 &= \sqrt{E + \xi} \left(\frac{RE'}{2} \int_1^{\mathfrak{h}} \frac{s^{1/2}(s-1)ds}{(Es + \xi)^{3/2}} - \frac{R'_0}{\sqrt{E + \xi^{\sharp}}} \right).
 \end{aligned}$$

Since $\frac{\nu}{(E\nu + \xi)^3} \leq \frac{4}{27E\xi^2}$ we find

$$\int_1^{\mathfrak{h}} \frac{\nu^{1/2}(\nu-1)d\nu}{(E\nu + \xi)^{3/2}} \leq \frac{2}{\xi\sqrt{27E}} \int_1^{\mathfrak{h}} (\nu-1) d\nu = \frac{(\mathfrak{h}-1)^2}{\xi\sqrt{27E}}$$

Lacking any other simplifying assumptions, we thus obtain strong criteria for local solvability:

Proposition 1.16. *System (1.13) is locally solvable at any point of \mathfrak{D}^{\pm} where $\delta \neq \sigma$ or where $2M \neq R$ if either of the following holds:*

- 1) $\text{sgn}E' \neq \text{sgn}R'_0$
- 2) $\left| \frac{E' \cdot (R_0 - R)^2}{2M\sqrt{27E}} \right| < \left| \frac{R'_0\sqrt{R_0}}{\sqrt{ER_0 + M}} \right|$

Indeed, given $r_0, t_0, M_0 > 0$ and smooth $E, R, R'_0 > 0$, the system (1.13) has on an open interval $I \ni z_0$ a unique solution satisfying

$$\vec{X}(z_0) = \begin{pmatrix} r_0 \\ t_0 \\ M_0 \end{pmatrix}.$$

2. DECOUPLED EQUATIONS: A CASE OF CONSTANT E

We consider the case of constant $E > 0$ in which we can rescale M and R to assume the case $E = 1$, retaining

$$(2.1) \quad \left(\frac{\dot{R}}{R} \right)^2 = \frac{2}{R^2} + \frac{2M}{R^3}$$

Here, equations (1.13) reduce to

$$(2.2) \quad \begin{aligned} \frac{dr}{dz} &= \frac{-\mathcal{G}AR}{\mathcal{F}} + \frac{R_z \cdot (1 - \frac{M}{R}\mathcal{G})\sqrt{3}}{(\sqrt{3} - \sigma\delta\sqrt{2 + 2M/R})\mathcal{F}} \\ \frac{dt}{dz} &= \frac{-\delta \cdot R_z}{\sqrt{3} - \sigma\delta\sqrt{2 + 2M/R}} \\ \frac{dM}{dz} &= RA + \frac{R_z \frac{M}{R} \sqrt{3}}{\sqrt{3} - \sigma\delta\sqrt{2 + 2M/R}} \end{aligned}$$

with $A = \frac{\sigma\sqrt{6}\sqrt{1+M/R}}{1+z}$.

For the remainder of the section we assume that $E, \sigma, \delta \equiv 1$ and denote by \mathcal{T}_1 the corresponding subset of \mathcal{T} . Then, $R(t, r) < R_0(r) \forall t < t_0$. And, for \mathfrak{h} and ξ as above, we obtain

$$\begin{aligned} \partial_M J &= -\frac{1}{2} \int_1^{\mathfrak{h}} \sqrt{\frac{\nu}{\nu + \xi}} \frac{1}{\nu + \xi} d\nu \\ \partial_{R_0} J &= \sqrt{\frac{1}{1 + \xi^{\mathfrak{h}}}}; \quad \partial_R J = -\sqrt{\frac{1}{1 + \xi}} \end{aligned}$$

with $\xi \geq \xi^\sharp$ and $\mathfrak{h} \geq 1$, so that the following hold:

$$\begin{aligned}
 (2.3) \quad 0 \leq \mathcal{J}_1(r, z, \xi) &\stackrel{\text{def}}{=} -\frac{\mathcal{G}}{\mathcal{F}} = \frac{\sqrt{1+\xi^\sharp}}{2R'_0} \int_1^{\mathfrak{h}} \sqrt{\frac{\nu}{\nu+\xi}} \frac{1}{\nu+\xi} d\nu \\
 &\leq \frac{\sqrt{\mathfrak{h}(1+\xi^\sharp)}}{R'_0} \left(\frac{1}{\sqrt{1+\xi}} - \frac{1}{\sqrt{\mathfrak{h}+\xi}} \right) \leq \frac{\sqrt{\mathfrak{h}}}{2R'_0(1+\xi)}; \\
 0 < \frac{1}{\mathcal{F}} &= \frac{1}{R'_0} \sqrt{\frac{1+\xi^\sharp}{1+\xi}} \stackrel{\text{def}}{=} \mathcal{J}_2(r, z, \xi) \leq 1/R'_0; \\
 0 < \frac{1-\xi\mathcal{G}}{\mathcal{F}} &= \mathcal{J}_2 + \xi\mathcal{J}_1 \leq \frac{1+\sqrt{\mathfrak{h}}/2}{R'_0}.
 \end{aligned}$$

Our change of variables leads to

$$\frac{d\xi}{dz} = \frac{dM}{dz}/R - R_z\xi/R$$

with $A = \frac{\sqrt{6}\sqrt{1+\xi}}{1+z}$ whereby the system (1.13) now reduces further to

$$\begin{aligned}
 (2.4) \quad \frac{dr}{dz} &= \frac{R\mathcal{J}_1\sqrt{6}\sqrt{1+\xi}}{1+z} + \frac{\sqrt{3}R_z \cdot (\mathcal{J}_2 + \xi\mathcal{J}_1)}{\sqrt{3} - \sqrt{2+2\xi}} \\
 \frac{dt}{dz} &= \frac{-R_z}{\sqrt{3} - \sqrt{2+2\xi}} \\
 \frac{d\xi}{dz} &= \frac{\sqrt{6}\sqrt{1+\xi}}{1+z} + \xi \frac{R_z}{R} \left(\frac{\sqrt{3}\sqrt{1+\xi}}{\sqrt{3} - \sqrt{2+2\xi}} \right).
 \end{aligned}$$

Here, we note that the equation for $\frac{d\xi}{dz}$ decouples from the others, allowing for ξ to be solved for explicitly in z . Then, with the solution to $\xi(z)$ in hand, both \mathcal{I}_1 and \mathcal{I}_2 depend only on z and r whereby the remaining equations are then decoupled.

We give estimates for the system (2.4) assuming uniform bounds on R , M/R , R_z , R_0 , and R'_0 . We suppose the following bounds hold for $0 < z_0 \leq z \leq z_1$ and

$0 < r, M, t$ on some compact sets (to be determined): $\xi \leq \xi^*$ with $|2\xi - 1| \geq \epsilon$
 > 0 ; $\rho_{min} \leq R \leq \rho_{max}$; $|R_z| \leq \lambda$; $1 < \mathfrak{h} \leq \mathfrak{h}^*$; and, $|R'_0| \geq \mathfrak{r} > 0$. Here, applying
(2.3)

$$(2.5) \quad \begin{aligned} \left| \frac{dt}{dz} \right| &\leq \frac{\lambda}{\sqrt{3} - \sqrt{2 + 2\xi^*}} = \frac{\lambda \cdot (\sqrt{3} + \sqrt{2 + 2\xi^*})}{\epsilon} \stackrel{\text{def}}{=} \mathfrak{M}_1 \\ \left| \frac{dr}{dz} \right| &\leq \sqrt{3} \frac{\rho_{max} \sqrt{\mathfrak{h}^*/2} + (1 + \sqrt{\mathfrak{h}^*/2}) \mathfrak{M}_1}{\mathfrak{r}} \stackrel{\text{def}}{=} \mathfrak{M}_2 \\ \left| \frac{d\xi}{dz} \right| &\leq \sqrt{3(1 + \xi^*)} (\sqrt{2} + \mathfrak{M}_1 \xi^* / \rho_{min}) \stackrel{\text{def}}{=} \mathfrak{M}_3 \end{aligned}$$

Let $\mathfrak{M} \stackrel{\text{def}}{=} \max_j \{\mathfrak{M}_j\}_{j=1}^3$ and suppose r_0, t_0 , and $M_0/R[z_0] \stackrel{\text{def}}{=} \xi_0 \neq 1/2$ satisfy
the restrictions on (r, t, ξ) for some $0 < \xi_0 < \xi^*$ as above with

$$(2.6) \quad \vec{X}_0 = \vec{X}(z_0) = \begin{pmatrix} r_0 \\ t_0 \\ M_0 \end{pmatrix}$$

For an interval I of the form $0 \leq z_0 \leq z \leq z_1$, the following now results from
standard theory of differential equations [4]:

Proposition 2.7. *For $z_0 \geq 0$, the system (2.2) is solvable on an interval of the
form $I = \{z | z_0 \leq z \leq z_1\}$ provided that the conditions (2.6) and (2.5) hold for
 \vec{X} in subset of \mathcal{T}_1 given by $|(\vec{X} - \vec{X}_0)_j| \leq b : j = 1, 2, 3$ for some constant b
 $< 1/\mathfrak{M}$. Here, a unique solution may be computed by the method of successive
approximations.*

Proof. We may apply Theorem 3.1, Chapt. 1 [4]: The conditions assure Lipschitz continuity of the right-hand sides of (2.4) and that both $z - z_0$ and $|\vec{X} - \vec{X}_0|/\mathfrak{M}$ are bounded above by $|z_1 - z_0|$, so that the result follows. \square

Recalling that we set $E \equiv 1$, we will suppose for the rest of the section that $R[z]$, $R'_0[z]$ are smooth and positive for $z > 0$. For some of our analysis below we will suppose also that

$$(2.8) \quad R[z] > Cz|R_z|.$$

holds on some real interval. We now present our estimates on $M[z]$ depending on $R[z]$ and initial conditions given by $\xi_0 \stackrel{\text{def}}{=} \xi(r(z_0), z_0)$.

Theorem 1. *Suppose that (2.8) holds on some interval $I = [z_0, z_1] \subset \mathbb{R}^+$. Then the following statements hold for some constants $0 < c_1 < 1/2 < c_2$, each depending on the choice of C :*

- 1) *If $0 < \xi_0 < 1/2$ and $R_z < 0$, then $M[z] \leq c_1 R[z]$ holds on I .*
- 2) *If $\xi_0 > 1/2$ and $R_z > 0$, then $M[z] \leq c_2 R[z]$ on I .*

Proof. Let us choose $C < 1/2$ and set $\Delta_\xi \stackrel{\text{def}}{=} \sqrt{3} - \sqrt{2 + 2\xi}$. In case 1) we use the estimate $1/\Delta_\xi \geq \sqrt{3}/(1 - 2\xi)$ for $0 < \xi < 1/2$ so that from (2.4)

$$\frac{d\xi}{dz} < \frac{\sqrt{6}}{z} \sqrt{1 + \xi} \left(1 - \frac{1}{C} \frac{\xi}{1 - 2\xi} \right).$$

Here, $\frac{d\xi}{dz} < 0$ for $1/2 > \xi > \xi_1^* \stackrel{\text{def}}{=} C/(2C + 1)$. Let $c_1 = \max\{\xi_0, \xi_1^*\}$.

In case 2) we note that $1/\Delta_\xi \leq -\sqrt{3}/(2\xi - 1)$ for $\xi > 1/2$. We find

$$\frac{d\xi}{dz} < \frac{\sqrt{6}}{z} \sqrt{1+\xi} \left(1 - \frac{1}{C} \frac{\xi}{2\xi - 1} \right)$$

and $\frac{d\xi}{dz} < 0$ for $1/2 < \xi < \xi_2^* \stackrel{\text{def}}{=} C/(2C - 1)$. Let $c_2 = \max\{\xi_0, \xi_2^*\}$. \square

Theorem 2. *Suppose $R_z < 0$ on $I = [z_0, \infty)$ with $z_0 > 0$ and $\xi_0 > 1/2$. Then, for $\rho \stackrel{\text{def}}{=} \sqrt{3/2}$, there are positive constants α , c_3 and c_4 so that the following holds on I :*

$$c_3 \left((R[z])^{-(\rho-1)} + R[z] \ln \left(\frac{1+z}{1+z_0} \right) \right) \leq M[z] \leq c_4 \left(\frac{1 + \ln(1+z)}{(R[z])^{(\rho-1/2)}} \right)^2$$

Proof. We first note that since $R_z/\Delta_\xi > 0$ on I , we find from (2.4) that $\frac{d\xi}{dz} > 0$.

Let us set $K_\xi \stackrel{\text{def}}{=} -\sqrt{3}\sqrt{1+\xi}/\Delta_\xi$, noting that $\rho \stackrel{\text{def}}{=} \sqrt{3/2} < K_\xi \leq K_{\xi_0}$ for $\xi > 1/2$ is decreasing as function of ξ and, in turn, also as a function of z . Recalling that $R_z < 0$, we find

$$\begin{aligned} \frac{d\xi}{dz} + \rho\xi \frac{R_z}{R[z]} &\geq \frac{d\xi}{dz} + K_\xi \xi \frac{R_z}{R[z]} \geq \sqrt{6} \frac{\sqrt{1+\xi_0}}{1+z}; \\ \frac{d}{dz} (\xi R^\rho) &\geq \sqrt{6} \sqrt{1+\xi_0} \frac{R^\rho[z]}{1+z}. \end{aligned}$$

Now, by the monotonicity of $R[z]$,

$$\begin{aligned} \xi[z] &\geq R^{-\rho}[z] \left(\xi_0 R^\rho[z_0] + \sqrt{6} \sqrt{1+\xi_0} \int_{z_0}^z \frac{R^\rho[s] ds}{1+s} \right) \\ &\geq R^{-\rho}[z] \left(\xi_0 R^\rho[z_0] + R^\rho[z] \sqrt{6} \sqrt{1+\xi_0} \int_{z_0}^z \frac{ds}{1+s} \right) \end{aligned}$$

After multiplying through by R , it is clear that we may choose $c_3 \leq \min\{\xi_0(R[z_0])^\rho, \sqrt{6}\sqrt{1+\xi_0}\}$. \blacksquare

Now, let us set $\rho_0 \stackrel{\text{def}}{=} K_{\xi_0}$ and $q_0 \stackrel{\text{def}}{=} (\xi_0 + 1)/\xi_0$. Then, for obvious substitution defining $\xi[s]$,

$$\begin{aligned} \frac{d\xi}{dz} + \rho_0 \xi \frac{R_z}{R[z]} &\leq \sqrt{6} \frac{\sqrt{1+\xi}}{1+z}; \\ \frac{d}{dz} (\xi R^{\rho_0}) &\leq \sqrt{6} \sqrt{1+\xi[z]} \frac{R^{\rho_0}[z]}{1+z} \\ \xi[z] &\leq R^{-\rho_0}[z] \left(\xi_0 R^{\rho_0}[z_0] + \sqrt{6} \int_{z_0}^z \frac{\sqrt{1+\xi[s]} R^{\rho_0}[s] ds}{1+s} \right) \\ &\leq R^{-\rho_0}[z] \left(\xi_0 R^{\rho_0}[z_0] + R^{\rho_0}[z_0] \sqrt{6} \sqrt{1+\xi[z]} \int_{z_0}^z \frac{ds}{1+s} \right) \\ \sqrt{\frac{\xi[z]}{q_0}} &< \frac{\xi[z]}{\sqrt{1+\xi[z]}} \leq R^{-\rho_0}[z] \left(\frac{R^{\rho_0}[z_0] \sqrt{\xi_0}}{\sqrt{1+\xi_0}} + \sqrt{6} R^{\rho_0}[z_0] \int_{z_0}^z \frac{ds}{1+s} \right) \\ \xi[z] &< q_0 R^{2\rho_0}[z_0] R^{-2\rho_0}[z] \left(1 + \sqrt{6} \int_{z_0}^z \frac{ds}{1+s} \right)^2 \end{aligned}$$

noting that $\xi^2/(1+\xi) \geq \xi/q_0$. Choosing $c_4 \geq q_0 6 R^{2\rho_0}[z_0]$, the result follows by multiplying through by R . \square

We see that Theorems 1 and 2 can apply for $R[z] = R_{\Omega_\Lambda}[z]$ (modulo a rescaling factor) as above for certain values of Ω_Λ : We denote by $I_{\Omega_\Lambda}^\pm$ the subset of $(0, \infty)$ for which $\pm R_z > 0$ and we replace C by C^\pm in the case that (2.8) holds, respectively.

Remark 2.9. For $R_{\Omega_\Lambda}[z]$ as in (0.7) we find that when $\Omega_\Lambda = 1$ there is to every interval of the form $(0, z_2)$, an associated C^+ depending on $z_2 > 0$. Moreover, for every $0 \leq \Omega_\Lambda < 1$ there is a $z_\Lambda > 0$ where for every positive z^\pm with $z^\pm \geq z_\Lambda$

there is a C^\pm associated to $(0, z^+)$ and (z^-, ∞) , respectively. [The singularities z_Λ will be discussed in further detail in Section 3.]

Proof. For $\Omega_\Lambda = 1$ we find $I_1^+ = (0, \infty)$ with $zR_z/R[z] = 1/(z+1)$. For $0 \leq \Omega_\Lambda < 1$, it is not difficult to show that $z/R[z]$ is bounded from below on $(0, \infty)$ by a positive constant, depending Ω_Λ . Therefore, the sign of $zR_z/R[z]$ depends on that of R_z . For $\Omega_\Lambda < 1$ we find that the sign of R_z is same as that of $(z+1)\mathcal{I}(z+1) - \int_1^{z+1} \mathcal{I}(y)dy$ which is a monotonically decreasing function of z with a unique positive root $z_\Lambda > 0$, depending on Ω_Λ . So, $I_{\Omega_\Lambda}^+ = (0, z_\Lambda)$ and $I_{\Omega_\Lambda}^- = (z_\Lambda, \infty)$. Hence, for z^\pm as above, there are positive constants C^\pm so that $zR_z(z)/R[z] > \pm C^\pm$ on intervals $(0, z^+)$ and (z^-, ∞) , respectively. \square

In a certain case of interest, we find that for certain initial conditions the growth of $M[z]$ roughly follows that of a power function for large z .

Corollary 2.10. *In the case of Theorem 2 we have for $R = R_{\Omega_\Lambda}$ with $0 \leq \Omega_\Lambda < 1$ that, given $M_0 > 2R[z_0] > 0$ and $z_0 > z_\Lambda$, for any $\alpha > 0$ there are positive constants k_1 and k_2 so that for $\rho = \sqrt{3/2}$,*

$$k_1 z^{\rho-1} \leq M[z] \leq k_2 z^{2\rho-1+\alpha}$$

on $I = [z_0, \infty)$.

Proof. It is not difficult to show that to any such Ω_Λ there are positive constants C_1 and C_2 so that

$$C_1/z < R_{\Omega_\Lambda}[z] < C_2/z$$

holds on I . The result immediately follows by Theorem 2. \square

We may also conclude

Corollary 2.11. *If either case 1) or 2) of Theorem 1 holds on $I = [z_0, z_1]$, then $r(z)$ and $t(z)$ are both solvable on I . Moreover, $r(z)$ is strictly increasing and $t(z)$ is strictly decreasing on I .*

Proof. We find that $\frac{dt}{dz}$ and $\frac{dr}{dz}$ are smooth functions of z since $\xi \neq 1/2$ is smooth. By inspection, we find that $\frac{dt}{dz} < 0$ on I so that, by our assumption on σ , we see for \mathfrak{h} as in (2.3) that $\mathfrak{h} \geq 1$, increasing with z . Then, $\mathcal{I}_2 + \xi\mathcal{I}_1 > 0$ for $z \in I$ and, hence, from (2.4) we see that $\frac{dr}{dz} > 0$ for $z \in I$. \square

We note finally that these results are consistent with physical interpretation where t is interpreted as "look-back" time from an observer at $r = 0$ with a (locally) expanding universe (c.f. [5, 12]).

3. STUDY OF SINGULARITIES, PART A: CRITICAL POINTS DEPENDING ON Ω_Λ

We now consider how singularities may depend on the parameter Ω_Λ for $R[z] = R_{\Omega_\Lambda}[z]$. As in Proposition 1.11, a singularity arises at $z = z_\Lambda$ where

$$(3.1) \quad [R_z]_{z=z_\Lambda} = \frac{(1+z_\Lambda) \cdot \mathcal{I}(1+z_\Lambda) - \int_1^{1+z_\Lambda} \mathcal{I}(y)dy}{(1+z_\Lambda)^2} = 0.$$

Proposition 3.2. *The values z_Λ satisfy $z_\Lambda \geq 1.25$, increasing as a continuous function of Ω_Λ in the domain $0 \leq \Omega_\Lambda < 1$. Moreover, there are positive constants c_1, c_2 and c_3 so that*

$$\left[c_1 \ln \left(\frac{1}{1 - \Omega_\Lambda} \right) + c_2 \right]^{1/4} \leq z_\Lambda + 1 \leq c_3 \frac{1}{1 - \Omega_\Lambda}$$

$\forall \Omega_\Lambda$. Hence, $z_\Lambda \rightarrow +\infty$ as $\Omega_\Lambda \rightarrow 1$.

Proof. It is not difficult to show from (3.1) that $z_\Lambda|_{\Omega_\Lambda=0} = 1.25$ and that $z_\Lambda > 0$

$\forall \Omega_\Lambda$. Now, let us set $q \stackrel{\text{def}}{=} 1 + z_\Lambda$ and note that (3.1) gives $q\mathcal{I}(q) = \int_1^q \mathcal{I}(y)dy$.

Implicit differentiation now gives

$$q \frac{dq}{d\Omega_\Lambda} \frac{\partial \mathcal{I}(q)}{\partial q} = \int_1^q \frac{\partial \mathcal{I}(y)}{\partial \Omega_\Lambda} dy - q \frac{\partial \mathcal{I}(q)}{\partial \Omega_\Lambda}.$$

Applying $q\mathcal{I}^3(q) = \mathcal{I}^2(q) \int_1^q \mathcal{I}(y)dy$ on the second term, right-hand side, we compute

$$Q(q) \frac{dq}{d\Omega_\Lambda} = \frac{1}{\mathcal{I}^3(q)} \int_1^q \mathcal{I}(y) K(y, q) dy$$

where $K(y, q) \stackrel{\text{def}}{=} \mathcal{I}^2(q)(q^3 - 1) - \mathcal{I}^2(y)(y^3 - 1)$ and $Q(q) \stackrel{\text{def}}{=} 3q^3(1 - \Omega_\Lambda)$. Since $\mathcal{I}^2(y)(y^3 - 1)$ is strictly increasing as a function of $y \geq 1$, we find $K(y, q) > 0$ for $1 \leq y < q$. Thus, $\frac{dq}{d\Omega_\Lambda} > 0 \forall \Omega_\Lambda$ and hence $q \geq 2.25 \forall \Omega_\Lambda$. For $k_1 \stackrel{\text{def}}{=} \frac{4}{3} \int_1^{2.25} \mathcal{I}(y) K(y, 2.25) dy$ we estimate

$$Q(q) \frac{dq}{d\Omega_\Lambda} \geq \frac{3k_1/4}{\mathcal{I}^3(q)} \geq 3k_1/4$$

so that

$$4 \int_{2.25}^q y^3 dy \geq k_1 \int_0^{\Omega_\Lambda} \frac{dx}{1-x},$$

and our choices of c_1 and c_2 are clear since

$$q^4 \geq k_1 \ln \left(\frac{1}{1 - \Omega_\Lambda} \right) + 2.25^4$$

Next we note that

$$q = \frac{1}{\mathcal{I}(q)} \int_1^q \mathcal{I}(y) dy \geq \sqrt{(1 - \Omega_\Lambda)q^3} \int_1^q \frac{dy}{\sqrt{1 + y^3}}$$

so that $\sqrt{q} \leq \frac{k_2}{\sqrt{1 - \Omega_\Lambda}}$ with $1/k_2 \stackrel{\text{def}}{=} \int_1^{2.25} \frac{dy}{\sqrt{1 + y^3}}$. We choose $c_3 = k_2^2$ and we are done. \square

With a broad range of values z_{Ω_Λ} , bound by the estimates of Proposition 3.2, one may expect difficulties in applying the present work to cosmological models - with singularities z_{Ω_Λ} well within observed redshift values [3, 5, 6]. However, some such singularities may conceivably be of type 0/0 if both R_z and $\sqrt{1 + 2E[z]} - \sqrt{2M[z]/R[z] + 2E[z]}$ were to have zeros of identical order, rendering the singularities, in some sense, removable. We demonstrate such a case in the next section.

4. STUDY OF SINGULARITIES, PART B: FRW MODEL

Using solutions from the well-known Freedman-Robertson-Walker model, we analyze our map $(E, D_L, R_0) \rightarrow (r, t, M)$ and study singularities of the system (1.13) and their dependence on Ω_Λ . We restrict the map as follows: We fix the function $E(r)$ and restrict $D_L(z)$ and $R_0(r)$ to certain one-parameter classes in

the pre-image space; and, we fix the function $M(r)$ in the image space. Here, we consider data given by $R[z] = R_{\Omega_\Lambda}[z]$ as in (0.7) and we set

$$(4.1) \quad E = \frac{r^2}{2}, M = \frac{r^3}{2}, R_0 = cM/E = cr$$

for parameter $c > 0$. Following [1], we have $R(r, t) = r \cdot a(t)$ where for some (real) parameter η with $k_c \stackrel{\text{def}}{=} \sqrt{c + c^2}$,

$$(4.2) \quad \begin{aligned} a(t) &= \frac{\cosh \eta - 1}{2} + (c \cosh \eta + k_c \sinh \eta) && \stackrel{\text{def}}{=} \mathfrak{F}_c(\eta) \\ t &= \frac{\sinh \eta - \eta}{\sqrt{2}} + \sqrt{2}(c \sinh \eta + k_c \cosh \eta) && \stackrel{\text{def}}{=} \mathfrak{G}_c(\eta) \end{aligned}$$

Here, η is known as "conformal time" which in our case depends on a and t by $\eta = \eta(t) = \int_{\sqrt{2}k_c}^t \frac{d\tau}{\sqrt{2}a(\tau)}$. We note that \mathfrak{F}_c and \mathfrak{G}_c are each invertible for η on an open interval containing 0. In particular, \mathfrak{F}_c is invertible for $\eta > -\text{arctanh}(2k_c/(1+2c))$ and \mathfrak{G}_c is invertible where $a > 0$, so that $a(t) = \mathfrak{F}_c \circ \mathfrak{G}_c^{-1}(t)$ indeed holds for t in a neighborhood containing k_c . Moreover, using (0.5) and setting $c = a(t_0)$ with $t_0 \stackrel{\text{def}}{=} t(z_0) = \sqrt{2}k_c$ for some $z_0 > 0$,

$$\frac{dt}{dz} = \frac{-a(t)}{(1+z)\dot{a}(t)}; \quad a[z] = a(t(z)) = c \frac{1+z_0}{1+z}.$$

Given $R[z]$, we find, indirectly, the resulting solutions of (1.13):

$$(4.3) \quad \begin{aligned} t(z) &= \mathfrak{G}_c(\mathfrak{F}_c^{-1}(a[z])) \\ r(z) &= R[z]/a[z] = \frac{\int_1^{1+z} \mathcal{I}(y)dy}{(1+z_0)c} \\ M[z] &= \frac{1}{2} \left(\frac{\int_1^{1+z} \mathcal{I}(y)dy}{(1+z_0)c} \right)^3 \end{aligned}$$

As for the relevance of this case to physical models, we note that the associated energy density $\rho[z]$ is a smooth function on $(0, \infty)$.

We are ready to state

Theorem 3. *For any given $0 \leq \Omega_\Lambda \leq 1$ and $z_0 > 0$ there is a smooth function $R_0(r)$ so that $E(r)$, $M(r)$ as in (4.1) and $R[z] = R_{\Omega_\Lambda}[z]$, the system (1.13) with initial conditions*

$$(4.4) \quad \vec{X}(z_0) = \begin{pmatrix} R[z_0]/c \\ \sqrt{2}k_c \\ (R[z_0]/c)^3/2 \end{pmatrix}$$

has a smooth solution \vec{X} on an open interval $I \ni z_0$.

Proof. For those η where the solutions (4.2) hold we also have $R' = a(t) > 0$ and $\dot{R}' = \frac{da}{dt} = \frac{da}{d\eta} / \frac{dt}{d\eta} > 0$. Since the initial conditions hold for $\eta = \mathfrak{F}_c^{-1}(a(z_0)) = 0$, (4.3) also holds for η in some interval containing 0. From continuity arguments we see there is also some open interval $I \ni z_0$ on which such solutions $\vec{X}(z)$ in turn hold. \square

We note that the above method provides no solutions for $r(z)$ and $M[z]$ in the case $R_0 \equiv 0$ unless more data is prescribed, such as asymptotic conditions for the ratio R/c in terms z and z_0 (c.f. Example A, p. 5 [6]). Moreover, we note that singularities may occur in the form $R = 2M$ and/or $\dot{a} = 0$ away from z_0 so that we may not arbitrarily extend the domain I of the solution via Proposition 1.14.

We may apply Proposition 1.14 in regards to uniqueness of solution: To rule out one type of singularity, we compute $E'\mathcal{G}R - \mathcal{F}$ via (1.15). First, we set $\xi = r/(2a(t))$ and $\xi_0 = r/(2c)$ and compute

$$\begin{aligned} \frac{E' \cdot (J_E - RJ_M)}{J_R} &= \sqrt{E + \xi} \frac{RE'}{2} \int_1^{c/a(t)} \frac{s^{1/2}(s-1)ds}{(Es + \xi)^{3/2}} \\ &= -\frac{1}{2} \int_{a(t)}^c \frac{\sqrt{\tau}(\tau - a(t))}{(\tau + 1)^{3/2}} d\tau \sqrt{\frac{a(t) + 1}{a(t)}} \leq 0 \end{aligned}$$

for $c, a(t) > 0$. We now compute,

$$\begin{aligned} \frac{R'_0 J_{R_0}}{J_R} &= -c \cdot \sqrt{\frac{R_0}{ER_0 + M}} / \sqrt{\frac{R}{ER + M}} \\ &= -c \cdot \sqrt{\frac{c}{c+1}} \sqrt{\frac{a(t) + 1}{a(t)}} \end{aligned}$$

which is strictly negative. Therefore, $E'\mathcal{G}R - \mathcal{F} < 0$ and we have ruled out case 2) of Proposition 1.10. Knowing also that $\dot{R}'[z]|_{z=z_0} \neq 0$ in this case we state

Theorem 4. *The solutions of Theorem 3 are unique for $z_0 \neq z_\Lambda$.*

The solutions (1.13) stand in glaring contrast to the result of Proposition 1.10: Indeed, we note that the right-hand sides of equations (0.4) and (0.5) under the

conditions of Theorem 3 have no positive singularities z_Λ as $E' + M'/R - MR'/R^2 = r + r/a > 0$; yet, we find that the determinant of \mathcal{U} in (1.9) vanishes at $z = z_\Lambda$. Since Theorem 3 applies in the case $z_0 = z_\Lambda$, one may suspect that these singularities are, in some sense, removable - so we shall see in remainder of this section.

We give specific cases, depending on R_0 , in which the solutions \vec{X} can be smoothly extended across singularities $z = z_\Lambda$. For such solutions to be valid, it suffices that $\dot{a}[z] > 0$ is smooth, that (3.1) holds, and that as in (1.12) $R[z_\Lambda] = 2M[z_\Lambda]$ (or perhaps as smooth extensions defined at z_Λ). Then,

$$R^2[z_\Lambda] = \frac{((1 + z_0)c)^3}{(1 + z_\Lambda)^3}$$

and, hence,

$$\mathcal{I}(1 + z_\Lambda) = R[z_\Lambda] = \frac{((1 + z_0)c)^{3/2}}{(1 + z_\Lambda)^{3/2}}.$$

From this we obtain the corresponding value of c by which we define

$$(4.5) \quad c_\Lambda \stackrel{\text{def}}{=} \frac{1 + z_\Lambda}{(1 + z_0)(\Omega_\Lambda + (1 - \Omega_\Lambda)(1 + z_\Lambda)^3)^{1/3}}.$$

We are ready to state

Theorem 5. *Under the hypothesis of Theorem 3 for every $z_0 > 0$ and $0 \leq \Omega_\Lambda < 1$ there is a smooth $R_0(r)$ for which the resulting solution $\vec{X}(z)$ with initial conditions (4.4) can be uniquely extended to be of class $C^\omega((0, \infty))$.*

Proof. We take $R_0(r) = c_\Lambda r$ for c_Λ as in (4.5). Using (4.2) and following the Chain Rule formula

$$(4.6) \quad \dot{a}(t(z)) = \frac{\frac{da[z]}{dz}}{\frac{d\mathfrak{G}_{c_\Lambda}(\eta(z))}{dz}} = \frac{\frac{da[z]}{dz}}{\sqrt{2}\mathfrak{F}_{c_\Lambda}(\eta(z))\frac{d\eta[z]}{dz}} = \frac{\frac{da[z]}{dz}}{\sqrt{2}a[z]\frac{d\eta[z]}{dz}},$$

it suffices to show that $\eta[z]$ is smooth and that $\frac{d\eta}{dz}$ is strictly positive on $(0, \infty)$.

To do this, we set

$$\eta[z] = - \int \frac{dt[s]}{\sqrt{2}a[s]},$$

with $dt[z] \stackrel{\text{def}}{=} \frac{dt(z)}{dz} dz$, and we proceed to analyze the integral. We may write

$$(4.7) \quad \frac{dt}{dz} = H(z) \frac{-R_z}{2M - R} = H(z) \frac{-R_z}{r(z)(r(z)^2 - a(z))}$$

for some real-valued function $H > 0$, analytic for $z > 0$. Here $r^2 - a$ is an increasing function which vanishes at z_Λ and is of the same sign as that of $-R_z$ $\forall z > 0$.

Now, we check the behavior of $\frac{dt}{dz}$ near the singularity, applying analyticity arguments as follows: Using (4.3) and (3.1) we compute

$$\begin{aligned} \left[\frac{d^2 R[z]}{dz^2} \right]_{z=z_\Lambda} &= -\frac{3(1-\Omega_\Lambda)}{2} (1+z_\Lambda) \mathcal{I}^3(1+z_\Lambda) < 0 \\ \left[\frac{dM[z]}{dz} \right]_{z=z_\Lambda} &= \frac{3\mathcal{I}(1+z_\Lambda)}{2(1+z_\Lambda)} \stackrel{\text{def}}{=} \mathfrak{M}_\Lambda > 0; \end{aligned}$$

and, in turn, we find that

$$2M[z] - R[z] = 2\mathfrak{M}_\Lambda \cdot (z - z_\Lambda) + \mathfrak{P}(z)(z - z_\Lambda)^2$$

for some analytic function \mathfrak{P} . Here, $\frac{d^2 R[z]}{dz^2} < 0$ on a neighborhood of z_Λ where R_z has a zero of order exactly 1. We therefore find that the following limit exists as we compute:

$$\begin{aligned} \lim_{z \rightarrow z_\Lambda} \frac{-\frac{dR}{dz}}{2M[z] - R[z]} &= \left[-\frac{\frac{d^2 R[z]}{dz^2}}{2\frac{dM[z]}{dz}} \right]_{z=z_\Lambda} \\ &= \frac{(1 - \Omega_\Lambda)(1 + z_\Lambda)\mathcal{I}(1 + z_\Lambda)}{2} > 0 \end{aligned}$$

We may conclude therefore that $\frac{-R_z}{2M[z] - R[z]}$ extends to an analytic, positive-valued function on $(0, \infty)$ and, hence, $C^\omega((0, \infty)) \ni \frac{d\eta[z]}{dz} > 0$. Therefore, $\dot{a}[z]$ is well-defined and is non-zero; and, moreover, $\eta[z]$ is of class $C^\omega((0, \infty))$. The uniqueness follows since Theorem 3 applies to any open interval not containing z_Λ \square

Remark 4.8. Following the calculations in the proof of Theorem 5, we note that any other choice of positive $c \neq c_\Lambda$ leads to a singularity of order one at $z = z_\Lambda$ for $\dot{R}'[z] = \dot{a}[z]$ as evident in (4.6) and (4.7). This gives singularities in equations (0.4), (0.5), and (1.2), and renders the resulting system (1.13) invalid at z_Λ .

Remark 4.9. Our FRW model is consistent with the construction of $R[z]$ as in [5] where $\frac{R_0}{R} = 1 + z$ with no prescribed value of c . Moreover, our choice of $c = c_\Lambda$ is optimal in assuring the largest possible domain of C^ω -solvability.

DISCUSSION

We make several concluding comments and a conjecture: First, we note that in the case of Theorem 5 the various right-hand sides of the system (1.13) can each

be written in the form $\mathcal{A}(z) + \mathcal{B}(z) \frac{R_z}{R[z] - 2M[z]}$ for smooth functions \mathcal{A} and \mathcal{B} . Thus, the arguments for the smooth extension of $\frac{dt}{dz}$ beyond the critical points z_Λ of $R[z]$ also apply to $\frac{dr}{dz}$ (also, of course with circular reasoning, to $\frac{dM}{dz}$). However, in the general mapping scheme (0.6), we have no way to predict the order of the zeros of $\frac{dM}{dz}$ nor any a priori justification to expect these singularities to be removable - not even as we fix our choice of $R[z] = R_{\Omega_\Lambda}[z]$.

Second, one may interpret the removability or non-existence of such singularities as indication of compatibility of the corresponding models as one imposes $R[z]$ on a model that prescribes E and R_0 . (Here the LTB model would be said to 'mimic' the given cosmological constant model, c.f. [2].) Applying such criteria to Remark 4.8 one does not expect every LTB model to be compatible with such a cosmological-constant model (at least not for z near z_Λ). However, from Theorem 5 we do find, as a check of our analysis, that the cosmological-constant models for $0 \leq \Omega_\Lambda < 1$ are each compatible with at least one LTB/FRW model: Our choice of R_0 identifies an optimal FRW model, in the sense of Remark 4.9.

Finally, one conjectures that these removable, 0/0-type singularities may yet lead to instability of numerical solutions of the system (1.13) (but here at certain finite z (!) c.f. §IV [6]). Such investigations are beyond the scope of the present work.

REFERENCES

- [1] H. Alnes, M. Amarzguoui, O. Gron, *Phys. Rev.*, D73, 083519, 2006.

- [2] A. Aguirre, Z. Haiman, Cosmological Constant or Intergalactic Dust? Constraints from the Cosmic Far-Infrared Background, *ApJ*, 532:28-36, 2000.
- [3] M.N. Celerier, Do we really see a cosmological constant in supernova data?, *A&A*, 2, 2008.
- [4] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
- [5] S. Carroll, W. Press, E. Turner, The cosmological constant, *ARAA*, 30, 1992.
- [6] D. Chung, A. Romano, Mapping luminosity-redshift relationship to LTB cosmology, *Phys. Rev.*, D74, 10103507, 2006.
- [7] K. Enqvist, Lemaitre-Tolman-Bondi model, *Gen. Rel. Grav.* 40:451-466, 2008.
- [8] J. Freiman, Lectures on Dark Energy and Cosmic Acceleration, *AIP Conf. Proc.*, vol. 1057, pp. 87-124, 2008.
- [9] J. Islam, *An Introduction to Mathematical Cosmology*, Cambridge University Press, 2002.
- [10] J. Kristian, R. Sachs, Observations in cosmology, *ApJ*, 143, 1966.
- [11] E.W. Kolb, S. Matarrese, A. Riotto, On cosmic acceleration without dark energy, *NewJ.Phys.* 8:322, 2006.
- [12] E. Kolb, M. Turner, *The Early Universe*, Addison Westly, 1990.
- [13] J.W. Mofit, Late-time Inhomogeneity and Acceleration Without Dark Energy, *JCAP*, 0605, 001 (2006)
- [14] *Gravitation*, C. Misner, K. Thorne, J. Wheeler, W.H. Freeman and Company, 1973.
- [15] H. Partovi, B. Mashhoon, Toward verification of large-scale homogeneity in cosmology, *ApJ* 276, 1984.
- [16] S. Rasanen, Backreaction in the Lemaitre-Tolman-Bondi model, *JCAP*, 0411:010, 2004

- [17] H. Stepani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 2003.
- [18] R. Wald, *General Relativity*, University of Chicago Press, 1984.
- [19] S. Weinberg, *Gravitation and Cosmology*, Wiley and Sons, Inc., 1972.

E-mail address: `winfield@madscitech.org`